J Glob Optim (2010) 48:183–208
DOI 10.1007/s10898-009-9486-z

Duality and optimality conditions for generalized equilibrium problems involving DC functions

N. Dinh · J. J. Strodiot · V. H. Nguyen

Received: 21 June 2009 / Accepted: 27 October 2009 / Published online: 8 November 2009

Abstract We consider a generalized equilibrium problem involving DC functions which is called (GEP). For this problem we establish two new dual formulations based on Toland-Fenchel-Lagrange duality for DC programming problems. The first one allows us to obtain a unified dual analysis for many interesting problems. So, this dual coincides with the dual problem proposed by Martinez-Legaz and Sosa (J Glob Optim 25:311–319, 2006) for equilibrium problems in the sense of Blum and Oettli. Furthermore it is equivalent to Mosco's dual problem (Mosco in J Math Anal Appl 40:202–206, 1972) when applied to a variational inequality problem. The second dual problem generalizes to our problem another dual scheme that has been recently introduced by Jacinto and Scheimberg (Optimization 57:795–805, 2008) for convex equilibrium problems. Through these schemes, as by products, we obtain new optimality conditions for (GEP) and also, gap functions for (GEP), which cover the ones in Antangerel et al. (J Oper Res 24:353–371, 2007, Pac J Optim 2:667–678, 2006) for variational inequalities and standard convex equilibrium problems. These results, in turn, when applied to DC and convex optimization problems with convex constraints (considered as special cases of (GEP)) lead to Toland-Fenchel-Lagrange duality for DC problems in Dinh et al. (Optimization 1–20, 2008, J Convex Anal 15:235–262, 2008), Fenchel-Lagrange and Lagrange dualities for convex problems as in Antangerel et al. (Pac J Optim 2:667–678, 2006), Bot and Wanka (Nonlinear Anal to appear), Jeyakumar et al. (Applied Mathematics research report AMR04/8, 2004). Besides, as consequences of the main results, we obtain some new optimality conditions for DC and convex problems.

This work was completed while the first author was visiting the Department of Mathematics of the University of Namur, Namur, Belgium in July-August 2007 and in August 2008.

N. Dinh
Department of Mathematics, International University, Vietnam National University,
Ho Chi Minh City, Vietnam

J. J. Strodiot (✉) · V. H. Nguyen
Department of Mathematics, University of Namur (FUNDP), Namur, Belgium
e-mail: jean-jacques.strodiot@fundp.ac.be

Keywords Equilibrium problems · Duality · Fenchel conjugation · DC functions · Variational inequalities

JEL Classification 49N15 · 58E35 · 90C26 · 90C46

1 Introduction

In this paper we consider the following generalized equilibrium problem of the model

$$(GEP)\quad \begin{cases} \text{Find } x \in K \text{ such that} \\ f(x, y) + \Psi(y) \geq \Psi(x) \text{ for all } y \in K, \end{cases}$$

where X is a locally convex Hausdorff topological space, K is a nonempty closed convex subset of X and $f : X \times X \to \mathbb{R} \cup \{+\infty\}$ and $\Psi : X \to \mathbb{R} \cup \{+\infty\}$ are functions satisfying:

(a) $f(x, x) = 0$ for all $x \in$ K;
(b) $f_x(\cdot) := f(x, \cdot)$ is proper, lower semi-continuous (l.s.c.), and convex for all $x \in K$;
(c) $\Psi = g - h$ where g, h: X $\to \mathbb{R} \cup \{+\infty\}$ are two proper, l.s.c., convex functions.

Here, by convention, we assume that $\infty - \infty = \infty + (-\infty) = +\infty$.

This problem is very general in the sense that it includes, as particular cases, many different problems as, for example, the problem of minimizing the difference of two convex functions, the mixed variational inequality problem, and when $\Psi = 0$, the Nash equilibrium problem in noncooperative games, the fixed point problem, the nonlinear complementarity problem and the vector optimization problem (see, for instance, Blum and Oettli [6] and the references quoted therein). The interest of such a general problem is that it unifies all these particular problems in a convenient way. Moreover, many results obtained for one of these problems can be extended with suitable modifications to the problem (GEP). However, the generalized equilibrium problem (GEP) is very important in itself. Indeed, it covers some important models in economics as, for example, the Nash-Cournot oligopolistic market equilibrium model with concave cost functions [26]. In this model, the function $f(x, y) = \langle F(x), y - x\rangle$ where F is affine and the function Ψ is a difference of two convex functions (a DC function in short).

Recently, duality results and optimality conditions have been obtained for equilibrium problems by Martinez-Legaz and Sosa [23] when $\Psi = 0$ and by Jacinto and Scheimberg [18] when Ψ is convex. Our aim in this paper is to obtain similar results but for the case where Ψ is a DC function. First, for each $x \in K$, we consider a DC optimization problem (P_x), which allows us to give a fixed point formulation of the solutions of (GEP). Then, we associate with each DC problem (P_x), a dual problem by using the Toland-Fenchel-Lagrange duality. This is the subject of Sect. 3 where we develop general duality and optimality results for a DC problem. In that section we also introduce a closedness condition, called (CC), that plays the role of a constraint qualification for these classes of problems. In Sect. 4 we use the duality for problems (P_x) to construct a first dual problem $(DGEP)$ associated with problem (GEP). When $\Psi = 0$, this dual reduces to the dual presented by Martinez-Legaz and Sosa in [23]. This dual problem also reduces to the ones introduced by Bigi, Castellani, and Kassey in [5] and by Mosco in [25] for variational inequality (VI) (see Sect. 7). First we prove weak and strong duality properties for these problems under the closedness condition (CC) which extended the corresponding results in [5, 23, 25]. We then establish necessary and sufficient optimality conditions for (GEP). These conditions, at the same time, give rise to the relationships between the solutions of problems (GEP) and $(DGEP)$. In particular, we prove that if the optimal value of the dual problem is zero (the primal problem (GEP) might not have any solution), then for any $\epsilon > 0$ the problem (GEP) admits ϵ-solutions. In the last part of this section we introduce another dual scheme that generalizes the dual presented by Jacinto and Scheimberg in [18] when Ψ is convex.

In Sect. 5 we propose gap functions related to the duality developed in the previous sections which extend the ones introduced in [1, 2] for variational inequalities and for equilibrium problems, while in Sects. 6 and 7 we show that our dual scheme allows us to find again well-known results when applied to special cases of problem (GEP) in [5, 11, 13, 14, 23, 25]. In particular, we develop in Sect. 6 the case of convex and DC optimization problems and find again several results established recently in [11, 13, 14]. Sect. 7 is devoted to the case of equilibrium problems in the sense of Blum and Oettli. First we prove that in the latter case the dual problem $(DGEP)$ coincides with Martinez-Legaz and Sosa's dual [23]. Then we show that in the particular case of variational inequality problems the dual problem $(DGEP)$ is equivalent to the dual introduced by Bigi, Castellani, and Kassey in [5] and by Mosco in [25].

2 Preliminaries

Let us recall some notations and properties useful in this paper. Let X be a locally convex Hausdorff topological vector space with its topological dual X^*, endowed with the weak*-topology.

The indicator function of a set $D \subset X$ is defined by $\delta_D(x) = 0$ if $x \in D$ and $\delta_D(x) = +\infty$ if $x \notin D$. Moreover, the support function σ_D is defined on X^* and is given by $\sigma_D(u) = \sup_{x \in D} u(x)$. When D^* is a subset of X^*, cl D^* stands for the closure of D^* with respect to the weak* topology in X^*.

Let $k : X \to \mathbb{R} \cup \{+\infty\}$ be a proper l.s.c., and convex function. The conjugate function of k, $k^* : X^* \to \mathbb{R} \cup \{+\infty\}$, is defined for all $v \in X^*$ by

$$k^*(v) = \sup\{\langle v, x\rangle - k(x) \mid x \in \text{ dom } k\},$$

where the domain of k is given by dom $k := \{x \in X \mid k(x) < +\infty\}$.

If $a \in$ dom k, then, following [19], we have

$$\text{epi } k^* = \bigcup_{\epsilon \geq 0} \{(v, v(a) + \epsilon - k(a)) \mid v \in \partial_\epsilon k(a)\}, \tag{1}$$

where, for a given $\epsilon \geq 0$, the ϵ- subdifferential of k at $a \in$ dom$k$, $\partial_\epsilon k(a)$, is defined as the possibly empty weak*-closed convex set

$$\partial_\epsilon k(a) = \{v \in X^* \ \mid \ k(x) - k(a) \geq (v, x - a) - \epsilon \text{ for all } x \in \text{dom } k\}.$$

If $\epsilon = 0$, then $\partial_\epsilon k(a)$ collapses to $\partial k(a)$, the usual subdifferential of k at a in the sense of convex analysis (for more details, see [28]).

Now let D be a convex subset of X and let $\varepsilon \geq 0$. The approximate normal cone at $a \in D$ is defined by

$$N_\varepsilon(D, a) = \{\, u \in X^* \mid u(x - a) \leq \varepsilon \text{ for all } x \in D \,\}.$$

When $\varepsilon = 0$, $N_\varepsilon(D, a)$ is the classical cone $N(D, a)$ of convex analysis. Moreover, it is easy to see that $N(D, a) = \partial\delta_D(a)$.

Following [8], it is worth noting that for two proper, l.s.c., and convex functions $k_1, k_2 : X \to \mathbb{R} \cup \{+\infty\}$, we have

$$\text{epi } (k_1 + k_2)^* = \text{cl}(\text{epi } k_1^* + \text{epi } k_2^*). \tag{2}$$

Moreover, epi k_1^* + epi k_2^* is weak*-closed if at least one of the functions k_1 and k_2 is continuous at some point of the domain of the other (see [8]).

Finally, we recall some results on DC programs which are useful for our study in the next sections. The first one is due to J.B. Hiriart-Urruty [17] and the second one to Toland [27].

Lemma 2.1 [17] *Let X be a locally convex Hausdorff topological vector space and let $F, G : X \to \mathbb{R} \cup \{+\infty\}$ be l.s.c., proper and convex functions. Then*

- (*i*) *A point $a \in X$ is a global minimizer of the problem* $\inf_{x \in X} \{F(x) - G(x)\}$ *if and only if for any $\varepsilon \geq 0$, $\partial_\varepsilon G(a) \subset \partial_\varepsilon F(a)$,*
- (*ii*) *If $a \in X$ is a local minimizer of* $\inf_{x \in X} \{F(x) - G(x)\}$*, then $\partial G(a) \subset \partial F(a)$.*

Lemma 2.2 [17, 27] *Let X be a locally convex Hausdorff topological vector space and let $F, G : X \to \mathbb{R} \cup \{+\infty\}$ be two proper and convex functions. If F is l.s.c. on X and $G^{**}(x) = G(x)$ for all $x \in X$, then*

$$\inf_{x \in X} \{F(x) - G(x)\} = \inf_{u \in X^*} \{G^*(u) - F^*(u)\}.$$

3 Duality of DC optimization problems

In this section we consider a general DC problem of model (Q) below. We establish optimality conditions and dual results for (Q) that will be the main tools for the establishment of the corresponding results for the generalized equilibrium problem (GEP) in the next sections. However the main results of this section may be of their own interest since they yield the standard Fenchel duality result for convex optimization problem (see Corollary 3.2 and also, [7, 8]), cover the subdifferential sum rule of convex functions established recently in [8] (see Corollary 3.1), and give rise to a new Farkas' lemma involving DC inequalities (see Corollary 3.4).

Consider the problem (GEP) defined in Sect. 1. For each $x \in K$, we associate with (GEP) the optimization problem

$$(P_x) \quad p(x) := \begin{cases} \inf \ f(x, y) + \Psi(y) \\ \text{s.t. } \ y \in K. \end{cases}$$

Since $f(x, x) = 0$ for all $x \in K$, the following result is straightforward from the definitions of problems (GEP) and (P_x).

Lemma 3.1 *A point $\bar{x} \in K$ is a solution of (GEP) if and only if $\bar{x}$ is a solution of $(P_{\bar{x}})$. In that case, $p(\bar{x}) = \Psi(\bar{x})$ and $\bar{x} \in Q(\bar{x})$ where $Q(x) := \arg\min \ (P_x)$, i.e., $\bar{x}$ is a fixed point of the mapping Q.*

It is worth mentioning that for each $x \in K$, (P_x) is the problem of minimizing the DC function (difference of two convex functions) $f_x(y) + g(y) - h(y)$ over the convex set K. These problems (P_x) are special cases of the following general DC problem

$$(Q) \quad \begin{cases} \inf \ F(y) + G(y) - H(y) \\ \text{s.t. } \ y \in K, \end{cases}$$

where X is a locally convex Hausdorff topological space, K is a closed convex subset of X, and $F, G, H : X \to \mathbb{R} \cup \{+\infty\}$ are proper, l.s.c., and convex functions. It is obvious that for each $x \in K$, (P_x) is of the model (Q) where f_x, g, and h play the roles of F, G, and H, respectively. We start with the following proposition which plays a key role in the study of (Q). This proposition may also have its own interest for it recovers the corresponding Theorem 1 in [8]. Several parts of its proof are similar to those of Theorem 3.1 in [13]. However, for the completeness of the paper, we give it in details.

From now on, the optimal value of problems (P_x) and (Q) are denoted $\nu(P_x)$ and $\nu(Q)$, respectively.

Proposition 3.1 (Conjugate and approximate subdifferential sum rules involving convex functions) *Assume that $U, V, T : X \to \mathbb{R} \cup \{+\infty\}$ are proper, l.s.c., and convex functions such that $\mathrm{dom} U \cap \mathrm{dom} V \cap \mathrm{dom} T \neq \emptyset$. The following statements are equivalent:*

(i) *epi $U^* +$ epi $V^* +$ epi T^* is weak*-closed,*
(ii) *For each $x^* \in X^*$,*

$$(U + V + T)^*(x^*) = \min_{u^*, v^* \in X^*} \{U^*(u^*) + V^*(v^*) + T^*(x^* - u^* - v^*)\} \tag{3}$$

(the infimum in the right-hand side is attained),
(iii) *For any $\bar{x} \in \mathrm{dom} U \cap \mathrm{dom} V \cap \mathrm{dom} T$ and each $\epsilon \geq 0$,*

$$\partial_\epsilon (U + V + T)(\bar{x}) = \bigcup_{\substack{\epsilon_1, \epsilon_2, \epsilon_3 \geq 0 \\ \epsilon_1 + \epsilon_2 + \epsilon_3 = \epsilon}} \left\{\partial_{\epsilon_1} U(\bar{x}) + \partial_{\epsilon_2} V(\bar{x}) + \partial_{\epsilon_3} T(\bar{x})\right\}. \tag{4}$$

Proof $[(i) \Rightarrow (ii)]$. Assume that (i) holds. Let $x^* \in X^*$. Then, for all $u^*, v^* \in X^*$, and $x \in X$, we have

$$\begin{aligned} U^*(u^*) \geq \langle u, x\rangle - U(x), \quad V^*(v^*) \geq \langle v^*, x\rangle - V(x), \\ T^*(x^* - u^* - v^*) \geq \langle x^* - u^* - v^*, x\rangle - T(x), \end{aligned} \tag{5}$$

which implies that for all $x \in X$,

$$U^*(u^*) + V^*(v^*) + T^*(x^* - u^* - v^*) \geq \langle x^*, x\rangle - (U + V + T)(x),$$

or, equivalently, that

$$\inf_{u^*, v^* \in X^*} \{U^*(u^*) + V^*(v^*) + T^*(x^* - u^* - v^*)\} \geq (U + V + T)^*(x^*). \tag{6}$$

If $x^* \notin \mathrm{dom}(U + V + T)^*$, then $(U + V + T)^*(x^*) = +\infty$ and (ii) holds. So, for proving the converse inequality in (3), it is sufficient to assume that $x^* \in \mathrm{dom}(U + V + T)^*$. Then we have

$$(x^*, (U + V + T)^*(x^*)) \in \text{epi } (U + V + T)^*. \tag{7}$$

On the other hand we observe that (see (2))

$$\begin{aligned} \text{epi } (U + V + T)^* &= \text{cl } \left(\text{epi } U^* + \text{epi } (V + T)^*\right) \\ &= \text{cl } \left(\text{epi } U^* + \text{cl } (\text{epi } V^* + \text{epi } T^*)\right) \\ &= \text{cl } \left(\text{epi } U^* + \text{epi } V^* + \text{epi } T^*\right). \end{aligned} \tag{8}$$

So, since (i) holds, we obtain that

$$\text{epi } (U + V + T)^* = \text{epi } U^* + \text{epi } V^* + \text{epi } T^*. \tag{9}$$

Combining now (7) and (9), we deduce that

$$(x^*, (U+V+T)^*(x^*)) = (u^*, r) + (v^*, s) + (w^*, t)$$

for some $(u^*, r) \in \text{epi } U^*$, $(v^*, s) \in \text{epi } V^*$, and $(w^*, t) \in \text{epi } T^*$. Consequently,

$$(U+V+T)^*(x^*) \geq U^*(u^*) + V^*(v^*) + T^*(x^* - u^* - v^*),$$

and (ii) follows. Finally, the infimum in (ii) is attained at some u^*, v^* such that $u^* \in \text{dom} U^*$, $v^* \in \text{dom} V^*$, and $x^* - u^* - v^* \in \text{dom} T^*$.

$[(ii) \Rightarrow (iii)]$. Assume (ii). Let $\bar{x} \in \text{dom} U \cap \text{dom} V \cap \text{dom} T$ and let $\epsilon \geq 0$. We firstly observe that the inclusion

$$\partial_\epsilon (U+V+T)(\bar{x}) \supset \bigcup_{\substack{\epsilon_1, \epsilon_2, \epsilon_3 \geq 0 \\ \epsilon_1+\epsilon_2+\epsilon_3=\epsilon}} \left\{ \partial_{\epsilon_1} U(\bar{x}) + \partial_{\epsilon_2} V(\bar{x}) + \partial_{\epsilon_3} T(\bar{x}) \right\}$$

in (iii) can be easily verified from the definition of approximate differentials in Sect. 1. For the converse inclusion, let $x^* \in \partial_\epsilon (U+V+T)(\bar{x})$. Since $\bar{x} \in \text{dom}(U+V+T)$, it follows from (1) that

$$\epsilon + \langle x^*, \bar{x} \rangle - U(\bar{x}) - V(\bar{x}) - T(\bar{x}) \geq (U+V+T)^*(x^*), \tag{10}$$

which shows that $x^* \in \text{dom}(U+V+T)^*$. Thanks to (ii), there exist $u^*, v^* \in X^*$, such that $u^* \in \text{dom} U^*$, $v^* \in \text{dom} V^*$, $x^* - u^* - v^* \in \text{dom} T^*$, and that $(U+V+T)^*(x^*) = U^*(u^*) + V^*(v^*) + T^*(x^* - u^* - v^*)$. So it follows from (10) that

$$\epsilon + \langle x^*, \bar{x} \rangle - U(\bar{x}) - V(\bar{x}) - T(\bar{x}) \geq U^*(u^*) + V^*(v^*) + T^*(x^* - u^* - v^*). \tag{11}$$

Now let $\epsilon_1 := U^*(u^*) - \langle u^*, \bar{x} \rangle + U(\bar{x})$. Then $\epsilon_1 \geq 0$ and $u^* \in \partial_{\epsilon_1} U(\bar{x})$. Similarly, we have $v^* \in \partial_{\epsilon_2} V(\bar{x})$ and $x^* - u^* - v^* \in \partial_{\epsilon_3'} T(\bar{x})$ where $\epsilon_2 := V^*(v^*) - \langle v^*, \bar{x} \rangle + V(\bar{x})$ and $\epsilon_3' := T^*(x^* - u^* - v^*) - \langle x^* - u^* - v^*, \bar{x} \rangle + T(\bar{x})$. Therefore,

$$x^* = u^* + v^* + (x^* - u^* - v^*) \in \partial_{\epsilon_1} U(\bar{x}) + \partial_{\epsilon_2} V(\bar{x}) + \partial_{\epsilon_3'} T(\bar{x}). \tag{12}$$

Note that (from (10) $0 \leq \epsilon_1 + \epsilon_2 + \epsilon_3' \leq \epsilon$. Let $\epsilon_3 := \epsilon - \epsilon_1 - \epsilon_2$. Then $\epsilon_3 \geq 0$ and $\epsilon_3' \leq \epsilon_3$, which entails that $\partial_{\epsilon_3'} T(\bar{x}) \subset \partial_{\epsilon_3} T(\bar{x})$. Combining this and (12), we obtain

$$x^* \in \partial_{\epsilon_1} U(\bar{x}) + \partial_{\epsilon_2} V(\bar{x}) + \partial_{\epsilon_3} T(\bar{x}),$$

which shows that

$$\partial_\epsilon (U+V+T)(\bar{x}) \subset \bigcup_{\substack{\epsilon_1, \epsilon_2, \epsilon_3 \geq 0 \\ \epsilon_1+\epsilon_2+\epsilon_3=\epsilon}} \left\{ \partial_{\epsilon_1} U(\bar{x}) + \partial_{\epsilon_2} V(\bar{x}) + \partial_{\epsilon_3} T(\bar{x}) \right\}.$$

$[(iii) \Rightarrow (i)]$. Assume that (iii) holds. Take $a \in \text{dom} U \cap \text{dom} V \cap \text{dom} T$ and $(x^*, r) \in \text{cl } \{\text{epi } U^* + \text{epi } V^* + \text{epi } T^*\}$. Then by (8), $(x^*, r) \in \text{epi } (U+V+T)^*$, and hence, it follows from (1) that there exists $\epsilon \geq 0$ such that $x^* \in \partial_\epsilon (U+V+T)(a)$ and

$$r = \langle x^*, a \rangle - (U+V+T)(a) + \epsilon \geq (U+V+T)^*(x^*). \tag{13}$$

By (iii), there exist $\epsilon_1, \epsilon_2, \epsilon_3 \geq 0$ and $u^*, v^*, w^* \in X^*$ such that $\epsilon_1 + \epsilon_2 + \epsilon_3 = \epsilon$, $x^* = u^* + v^* + w^*$, and $u^* \in \partial_{\epsilon_1} U(a)$, $v^* \in \partial_{\epsilon_2} V(a)$, $w^* \in \partial_{\epsilon_3} T(a)$. Again, by (1),

$$(u^*, \langle u^*, a \rangle + \epsilon_1 - U(a)) \in \text{epi } U^*,$$
$$(v^*, \langle v^*, a \rangle + \epsilon_2 - V(a)) \in \text{epi } V^*,$$
$$(w^*, \langle w^*, a \rangle + \epsilon_3 - T(a)) \in \text{epi } T^*,$$

which, in turn, implies that

$$(x^*, x^*(a) + \epsilon - (U + V + T)(a)) \in \text{epi } U^* + \text{epi } V^* + \text{epi } T^*.$$

The last inclusion and (13) entail that $(x^*, r) \in \text{epi } U^* + \text{epi } V^* + \text{epi } T^*$, which proves (i). □

Remark 3.1 Concerning statement (ii), it is worth noting that if $x^* \in \text{dom}(U+V+T)^*$, then there exist $u^* \in \text{dom}U^*$, $v^* \in \text{dom}V^*$ such that $x^* - u^* - v^* \in \text{dom}T^*$ and the infimum in the right hand side of (3) is attained at u^*, v^*. Otherwise, i.e., when $x^* \notin \text{dom}(U+V+T)^*$, for arbitrary $u^*, v^* \in X^*$, one has

$$U^*(u^*) + V^*(v^*) + T^*(x^* - u^* - v^*) = +\infty.$$

The following Corollary is useful for the study of Problem (P_x). It recovers Corollary 1 in [8] when one of the functions U, V, and T is a zero constant function.

Corollary 3.1 (Subdifferential sum rule involving convex functions) *Assume that $U, V, T : X \to \mathbb{R} \cup \{+\infty\}$ are proper, l.s.c., and convex functions. If epi U^* + epi V^* + epi T^* is weak*-closed, then for any $\bar{x} \in \text{dom}U \cap \text{dom}V \cap \text{dom}T$,*

$$\partial(U + V + T)(\bar{x}) = \partial U(\bar{x}) + \partial V(\bar{x}) + \partial T(\bar{x}).$$

Proof This is a direct consequence of Proposition 3.1. □

Theorem 3.1 (Optimality Condition for (Q)) *For Problem (Q), assume that epi F^* + epi G^* + epi δ_K^* is weak*-closed. Then*

(i) *$\bar{x} \in K$ is a global solution of (Q) if and only if for any $\epsilon \geq 0$*

$$\partial_\epsilon H(\bar{x}) \subset \bigcup_{\substack{\epsilon_1,\epsilon_2,\epsilon_3 \geq 0 \\ \epsilon_1+\epsilon_2+\epsilon_3=\epsilon}} \left\{\partial_{\epsilon_1} F(\bar{x}) + \partial_{\epsilon_2} G(\bar{x}) + N_{\epsilon_3}(K, \bar{x})\right\}.$$

(ii) *If $\bar{x} \in K$ is a local solution of (Q), then*

$$\partial H(\bar{x}) \subset \partial F(\bar{x}) + \partial G(\bar{x}) + N(K, \bar{x}).$$

Moreover, if $\bar{x}$ is a local solution of (Q), then for any $x^* \in \partial H(\bar{x})$, there exist $u^* \in \text{dom}F^*$, $v^* \in \text{dom}V^*$ such that $x^* - u^* - v^* \in \text{dom}\delta_K^*$ and

$$H^*(x^*) - F^*(u^*) - G^*(v^*) - \delta_K^*(x^* - u^* - v^*) = F(\bar{x}) + G(\bar{x}) - H(\bar{x}).$$

Proof Note that problem (Q) is equivalent to the minimization problem

$$\inf_{x \in X} \left[(F + G + \delta_K)(x) - H(x)\right].$$

Hence by Lemma 2.1, $\bar{x} \in K$ is a global solution of (Q) if and only if for any $\varepsilon \geq 0$,

$$\partial_\varepsilon H(\bar{x}) \subset \partial_\varepsilon (F + G + \delta_K)(\bar{x}).$$

Thus, (i) follows from this and Proposition 3.1.
Similarly, (ii) follows from Lemma 2.1 and Corollary 3.1. □

Theorem 3.2 (Duality for (Q)). *For Problem (Q), we have*

(i) $$\nu(Q) \geq \inf_{x^* \in X^*} \left\{ \sup_{u^*, v^* \in X^*} \left[H^*(x^*) - F^*(u^*) - G^*(v^*) - \delta_K^*(x^* - u^* - v^*)\right] \right\}.$$

(ii) *If epi* $F^* +$ *epi* $G^* +$ *epi* δ_K^* *is weak***-closed, then*

$$\nu(Q) = \inf_{x^* \in X^*} \left\{ \max_{u^*, v^* \in X^*} \left[H^*(x^*) - F^*(u^*) - G^*(v^*) - \delta_K^*(x^* - u^* - v^*) \right] \right\} \quad (14)$$

Proof The inequality in (i) follows easily from the definition of conjugate functions and from that $H^{**} = H$. In fact, for any $x^*, u^*, v^* \in X^*$ and any $y \in K$, by definition of conjugate functions, we have

$$\begin{aligned} H^*(x^*) \; - \; & F^*(u^*) - G^*(v^*) - \delta_K^*(x^* - u^* - v^*) \\ & \leq H^*(x^*) + \langle -u^*, y \rangle + F(y) + \langle -v^*, y \rangle + G(y) + \langle -x^* + u^* + v^*, y \rangle \\ & \quad + \delta_K(y) \leq H^*(x^*) - \langle x^*, y \rangle + F(y) + G(y). \end{aligned}$$

This yields for all $y \in K$ (note that $H^{**}(y) = H(y)$),

$$\begin{aligned} \inf\nolimits_{x^* \in X^*} \sup\nolimits_{u^*, v^* \in X^*} & \left\{ H^*(x^*) - F^*(u^*) - G^*(v^*) - \delta_K^*(x^* - u^* - v^*) \right\} \\ & \leq \inf\nolimits_{x^* \in X^*} [H^*(x^*) - \langle x^*, y \rangle] + F(y) + G(y) \\ & = -H^{**}(y) + F(y) + G(y) \\ & = F(y) + G(y) - H(y), \end{aligned}$$

which proves (i) since the last inequality holds for any $y \in K$.

For the proof of (ii), note that

$$\nu(Q) = \inf_{y \in K} [F(y) + G(y) - H(y)] = \inf_{y \in X} [(F + G + \delta_K)(y) - H(y)].$$

Hence, by Toland's duality theorem (Lemma 2.2),

$$\nu(Q) = \inf_{y \in X} [(F + G + \delta_K)(y) - H(y)] = \inf_{x^* \in X^*} \{H^*(x^*) - (F + G + \delta_K)^*(x^*)\}. \quad (15)$$

Since epi $F^* +$ epi $G^* +$ epi δ_K^* is weak*-closed, it follows from Proposition 3.1 that

$$(F + G + \delta_K)^*(x^*) = \min_{u^*, v^* \in X^*} \{F^*(u^*) + G^*(v^*) + \delta_K^*(x^* - u^* - v^*)\}.$$

Combining this and (15), we obtain that

$$\nu(Q) = \inf_{x^* \in X^*} \max_{u^*, v^* \in X^*} \{H(x^*) - F^*(u^*) - G^*(v^*) - \delta_K^*(x^* - u^* - v^*)\},$$

which completes the proof. □

Now the general Fenchel duality result for convex problems in infinite dimensional spaces (see [7, 8] and the references quoted therein) follows from the previous proposition as shown in the next corollary.

Corollary 3.2 [8] (Fenchel Duality for the sum of convex functions). *Assume that* $F, G : X \to \mathbb{R} \cup \{+\infty\}$ *are proper, l.s.c., and convex functions. Assume further that epi* $F^* +$ *epi* G^* *is weak***-closed. Then*

$$\inf_{y \in X} [F(y) + G(y)] = \max_{u^* \in X^*} [-F^*(-u^*) - G^*(u^*)].$$

Proof Let $K = X$. Then $\delta_K \equiv 0$ on X and thus, dom $\delta_K^* = \{0\}$ and epi $\delta_K^* = \{0\} \times [0, +\infty)$. The conclusion now follows directly from Theorem 3.2 where $H \equiv 0$ and $K = X$ since

$$\begin{aligned} \text{epi } F^* + \text{epi } G^* + \text{epi } \delta_K^* & = \text{epi} F^* + \text{epi} G^* + \{0\} \times [0, \infty) \\ & = \text{epi} F^* + \text{epi} G^*. \end{aligned}$$

□

The following corollary is a direct consequence of Theorem 3.2.

Corollary 3.3 (Fenchel Duality for convex problems). *For problem (Q), assume that $H \equiv 0$ and $G \equiv 0$. Assume further that epi $F^* +$ epi δ_K^* is weak*-closed. Then*

$$\inf(Q) = \max_{u^* \in X^*} \left[-F^*(u^*) - \delta_K^*(-u^*) \right].$$

A Farkas lemma of the same dual type as in [14], and involving DC functions, now follows from Theorem 3.2.

Corollary 3.4 (Farkas Lemma). *Let $\alpha \in \mathbb{R}$. Assume that epiF^* + epiG^* + epi δ_K^* is weak*-closed. Then the following statements are equivalent:*

(i) *For all $y \in K$, $F(y) + G(y) - H(y) \geq \alpha$,*
(ii) *For each $x^* \in X^*$, there exist $u^*, v^* \in X^*$ such that*

$$H^*(x^*) - F^*(u^*) - G^*(v^*) - \delta_K^*(x^* - u^* - v^*) \geq \alpha.$$

Proof It is clear that (i) is equivalent to

$$\inf_{y \in K} [F(y) + G(y) - H(y)] \geq \alpha.$$

By Theorem 3.2, the last inequality is equivalent to

$$\inf_{x^* \in X^*} \max_{u^*, v^* \in X^*} \{H(x^*) - F^*(u^*) - G^*(v^*) - \delta_K^*(x^* - u^* - v^*)\} \geq \alpha,$$

which, in turn, is equivalent to (ii). □

We now come back to problem (P_x) associated with (GEP). The duality results for the general problem (Q) give rise to the corresponding ones for (P_x) as shown below. But first, we introduce the definition of a constraint qualification called closedness property for (P_x).

Definition 3.1 (*Closedness Condition*). Let $x \in K$. If the set

$$(CC) \qquad \text{epi } f_x^* + \text{epi } g^* + \text{epi } \delta_K^*$$

is weak*-closed in the dual space of $X \times \mathbb{R}$, then problem (P_x) is said to satisfy the closedness condition, ((CC) in short), or equivalently, that problem (GEP) satisfies the closedness condition (CC) at x.

It is worth observing that the dual form constraint qualification of the type (CC) seems to be used for the first time in [8, 20]. This condition is weaker than several qualification conditions known in the literature such as generalized Slater conditions and interior-type conditions. It was successfully used for establishing optimality conditions, duality, stability of convex programming problems [8, 20], convex infinite programs [10, 15], DC problems with convex constraints [13, 14], and DC infinite programs with parameters [12]. It was also used to study the variational inequalities and equilibrium problems (see [1, 2]). In this paper this condition plays a key role in the study of duality and other topics in the next sections.

For $x \in K$, $x^*, u^*, v^* \in X^*$, set

$$L(x, x^*, u^*, v^*) = h^*(x^*) - f_x^*(u^*) - g^*(v^*) - \delta_K^*(x^* - u^* - v^*). \tag{16}$$

As a consequence of Theorem 3.2, we obtain the following corollary.

Corollary 3.5 (Duality for (P_x)). *Let $x \in K$. Then*

(i) $\nu(P_x) \geq \inf_{x^* \in X^*} \left\{ \max_{u^*, v^* \in X^*} L(x, x^*, u^*, v^*) \right\}$.

(ii) *If* (P_x) *satisfies* (CC), *then*

$$\nu(P_x) = \inf_{x^* \in X^*} \left\{ \max_{u^*, v^* \in X^*} L(x, x^*, u^*, v^*) \right\}. \tag{17}$$

Moreover, in the case (ii), *if* $h \equiv 0$ *then*

$$\nu(P_x) = \max_{u^*, v^* \in X^*} L(x, x^*, u^*, v^*). \tag{18}$$

(*the "sup" in the right hand-side is attained*).

Proof This is a direct consequence of Theorem 3.2 where f_x, g, and h play the roles of F, G, and H, respectively. □

4 Duality and optimality conditions for (GEP)

In this section we introduce a dual problem $(DGEP)$ associated with problem (GEP). We give weak and strong duality results and optimality conditions for (GEP) as well. The latter, at the same time, shows the relationships between the solutions of (GEP) and those of $(DGEP)$. Besides, it is shown that if the primal problem (GEP) possesses an optimal solution then the value of the dual problem $(DGEP)$ is zero, i.e., $\nu(DGEP) = 0$. However, the converse is not true. In such a situation, i.e., when the optimal value of the dual problem $(DGEP)$ is zero, we prove that for any $\epsilon > 0$, the problem (GEP) possesses at least an ϵ-solution. As it is proven in Sect. 7, when $\Psi = 0$, the dual problem $(DGEP)$ coincides with the dual problem defined by Martinez-Legaz and Sosa in [23], and when $h \equiv 0$ and $f(x, y) = \langle F(x), y - x \rangle$ where F is an operator from X to X^*, the dual problem $(DGEP)$ is equivalent to the dual problem for variational inequality problems in the sense of Mosco [25]. Finally, to end this section, we generalize to our problem another dual scheme that covers the one recently introduced by Jacinto and Scheimberg in [18] for problem (GEP) for the case where $h \equiv 0$.

First let us recall that for problem (GEP), each $x \in K$ is associated to an optimization problem (P_x):

$$p(x) = \inf_{y \in K} [f_x(y) + \Psi(y)].$$

From this definition and from Lemma 3.1, we can conclude that

- $p(x) \leq \Psi(x)$ for all $x \in K$, and
- $\bar{x} \in K$ is a solution of (GEP) if and only if $p(\bar{x}) = \Psi(\bar{x})$.

Definition 4.1 (*Local solutions of* (GEP)). A point $\bar{x} \in K$ is called a *local solution* of (GEP) if there exists a neighborhood U of $\bar{x}$ such that

$$f(\bar{x}, y) + \Psi(y) \geq \Psi(\bar{x}) \quad \text{for all } y \in U \cap K.$$

It is obvious that $\bar{x} \in K$ is a local solution of (GEP) if and only if it is a local solution of $(P_{\bar{x}})$. Furthermore, any global solution of (GPE) is also a local solution of this problem. So the problem of finding (local/global) solutions of (GEP) reduces to the one of finding (local/global) solutions of the optimization problem

$$\text{(P)} \qquad \max_{x \in K} [p(x) - \Psi(x)],$$

or, equivalently, to the problem

$$\max_{x\in K}\left[\inf_{y\in K}\{f_x(y)-\Psi(y)\}-\Psi(x)\right].$$

By the weak duality property for (P_x) (Corollary 3.5 (i)), we obtain

$$\nu(P)=\max_{x\in K}[p(x)-\Psi(x)]\geq \max_{x\in K}\left[\inf_{x^*\in X^*}\max_{u^*,v^*\in X^*}L(x,x^*,u^*,v^*)-\Psi(x)\right].$$

The last inequality stimulates us to define the problem in the right hand side to be the dual problem of (GEP). Concretely, the problem $(DGEP)$ defined by

$$(DGEP)\qquad \max_{x\in K}\left[\inf_{x^*\in X^*}\max_{u^*,v^*\in X^*}L(x,x^*,u^*,v^*)-\Psi(x)\right]$$

is called the dual problem of the generalized equilibrium problem (GEP).

It is worth mentioning that this dual problem collapses to the one introduced by Martinez-Legaz and Sosa in [23] when $h\equiv 0$ and $g\equiv 0$. When $h\equiv 0$ and $f(x,y)=\langle F(x),y-x\rangle$ where F is an operator from X to X^*, the dual problem $(DGEP)$ is equivalent to the dual problem for variational inequality problems in the sense of Mosco [25] and that of Bigi, Castellani, and Kassey in [5].

Definition 4.2 (*Solution of* $(DGEP)$) A solution of the dual problem is a point $\bar{x}\in K$ such that for any $x^*\in \operatorname{dom} h^*$, there exist $u^*,v^*\in X^*$ such that the following equality holds

$$L(\bar{x},x^*,u^*,v^*)=g(\bar{x})-h(\bar{x}). \tag{19}$$

The point $\bar{x}$ is said to be a weak solution of the dual if for any $x^*\in\partial h(\bar{x})$, there exist $u^*,v^*\in X^*$ such that the equality (19) holds.

Remark 4.1 Note that the inequality

$$L(\bar{x},x^*,u^*,v^*)\leq g(\bar{x})-h(\bar{x})$$

is always true. Indeed, using the same argument as in the first part of the proof of Theorem 3.2, we get for all $y\in K$,

$$\begin{aligned}L(\bar{x},x^*,u^*,v^*)&=h^*(x^*)-f^*_{\bar{x}}(u^*)-g^*(v^*)-\delta^*_K(x^*-u^*-v^*)\\&\leq h^*(x^*)-\langle x^*,y\rangle+g(y)+f_{\bar{x}}(y)\\&\leq -h(y)+g(y)+f_{\bar{x}}(y).\end{aligned}$$

Taking $y=\bar{x}$, we obtain the desired inequality. Therefore, the equality in the definition (19) is equivalent to

$$L(\bar{x},x^*,u^*,v^*)\geq g(\bar{x})-h(\bar{x}).$$

Furthermore, it is easy to see that $\nu(DGEP)\leq 0$.

The weak and strong duality results are given in the next theorem.

Theorem 4.1 (Weak and strong duality for (GEP)) *For problem (GEP), the following properties hold:*

(i) $\nu(P)\geq \nu(DGEP)$.
(ii) *If for each $x\in K$, the closedness condition (CC) holds, then*

$$\nu(P)=\nu(DGEP).$$

Proof (i) is obvious. To prove (ii), note that if (CC) holds, then problem (P_x) enjoys the strong duality property for each x, i.e.,

$$p(x) = \inf_{x^* \in X^*} \max_{u^*, v^* \in X^*} L(x, x^*, u^*, v^*).$$

Hence, $\nu(P) = \nu(DGEP)$. □

Remark 4.2 Note that if the problem (GEP) has a solution then $\nu(P) = 0$. In Theorem 4.1, the equality (ii) holds provided that (CC) holds for every $x \in K$. But even in this case, the values of each side in this equality might not be zero if (GEP) has no solution as shown in the following simple example.

Example 4.1 [26] Consider the generalized equilibrium problem $(P1)$ of finding $x \in K := [-1, 1]$ such that

$$\langle x, y - x\rangle - y^2 \geq -x^2 \ \text{ for all } y \in K.$$

This problem is of the model (GEP) where $\Psi(x) = h(x) = -x^2$ is a concave function, $g \equiv 0$, and $f(x, y) = \langle x, y - x\rangle$. It is easy to see that (CC) holds for all $x \in [-1, 1]$ and $\nu(P1) = \nu(DP1) = -1 \neq 0$ (where$(DP1)$ is the dual problem of $(P1)$). It is necessary to emphasize that $(P1)$ has no solution.

We now establish the relationship between the solutions of (GEP) and those of $(DGEP)$, and we derive at the same time, optimality conditions for (GEP). First we consider the local solutions of (GEP).

Theorem 4.2 *Let $\bar{x} \in K$. For the problem (GEP), assume that the closedness condition (CC) holds at $\bar{x}$ and that $\partial h(\bar{x}) \neq \emptyset$. If $\bar{x}$ is a local solution of (GEP), then for any $x^* \in \partial h(\bar{x})$ there exist $u^* \in dom f^*_{\bar{x}}$, $v^* \in dom g^*$ such that*

$$L(\bar{x}, x^*, u^*, v^*) = g(\bar{x}) - h(\bar{x}). \tag{20}$$

In particular, $\bar{x}$ is a weak solution of $(DGEP)$.

Proof Let $\bar{x}$ be a local solution of (GEP). Then $\bar{x}$ is a local solution of the DC program

$$\inf_{y \in K} [f_{\bar{x}}(y) + g(y) - h(y)].$$

Since (CC) holds for (P_x), it follows from [27] and the subdifferential sum rule, Corollary 3.1, that

$$\begin{aligned} \partial h(\bar{x}) &\subset \partial(f_{\bar{x}} + g + \delta_K)(\bar{x}) \\ &\subset \partial f_{\bar{x}}(\bar{x}) + \partial g(\bar{x}) + N_K(\bar{x}). \end{aligned} \tag{21}$$

Let $x^* \in \partial h(\bar{x})$ (this set is non-empty by assumption). By (21), there exist $u^* \in \partial f_{\bar{x}}(\bar{x})$, $v^* \in \partial g(\bar{x})$, and $w^* \in N_K(\bar{x}) = \partial\delta_K(\bar{x})$ such that $x^* = u^* + v^* + w^*$, which give rise to

$$\begin{aligned} &f^*_{\bar{x}}(u^*) = f^*_{\bar{x}}(u^*) + f_{\bar{x}}(\bar{x}) = \langle u^*, \bar{x}\rangle, \\ &g^*(v^*) + g(\bar{x}) = \langle v^*, \bar{x}\rangle, \quad h^*(x^*) + h(\bar{x}) = \langle x^*, \bar{x}\rangle, \\ &\delta^*_K(x^* - u^* - v^*) = \delta^*_K(x^* - u^* - v^*) + \delta_K(\bar{x}) = \langle x^* - u^* - v^*, \bar{x}\rangle. \end{aligned}$$

Combining these equalities, we get

$$h^*(x^*) - f^*_x(u^*) - g^*(v^*) - \delta^*_K(x^* - u^* - v^*) = g(\bar{x}) - h(\bar{x}),$$

and the theorem is proven. □

Now we consider the global solutions of (GEP).

Theorem 4.3 *Let $\bar{x} \in K$. For problem (GEP), assume that the closedness condition (CC) holds at $\bar{x}$. If $\bar{x}$ is a global solution of (GEP), then for each $x^* \in X^*$, there exist $u^*, v^* \in X^*$ satisfying*

$$L(\bar{x}, x^*, u^*, v^*) = g(\bar{x}) - h(\bar{x}). \tag{22}$$

In this case, $\bar{x}$ is a solution of the dual problem $(DGEP)$ and $\nu(DGEP) = 0$.

Proof Assume that $\bar{x}$ is a solution of (GEP). Then by Lemma 3.1, $\bar{x}$ solves $(P_{\bar{x}})$, which means that $\nu(P_{\bar{x}}) = g(\bar{x}) - h(\bar{x})$. It now follows from Corollary 3.5 (ii) that for each $x^* \in X^*$,

$$\nu(P_{\bar{x}}) = g(\bar{x}) - h(\bar{x}) \leq \max_{u^*, v^* \in X^*} L(\bar{x}, x^*, u^*, v^*),$$

which implies the existence of $u^*, v^* \in X^*$ such that

$$g(\bar{x}) - h(\bar{x}) \leq L(\bar{x}, x^*, u^*, v^*).$$

The reverse inequality being always true (see Remark 4.1), the equality (22) is proved.

The previous argument also shows that $\bar{x}$ is a solution of the dual problem $(DGEP)$. Moreover, in this case, since $p(\bar{x}) = \Psi(\bar{x})$ and (CC) holds at $\bar{x}$, by the strong duality for $(P_{\bar{x}})$ (Corollary 3.5),

$$p(\bar{x}) - \Psi(\bar{x}) = \inf_{x^* \in X^*} \left\{ \max_{u^*, v^* \in X^*} L(x, x^*, u^*, v^*) \right\} = 0,$$

and hence, $\nu(DGEP) \geq 0$, which together with the obvious inequality $\nu(DGEP) \leq 0$ gives $\nu(DGEP) = 0$. □

We now prove the converse of Theorem 4.3, which also gives a sufficient optimality condition for (GEP).

Theorem 4.4 *Let $\bar{x} \in K$. If for any $x^* \in domh^*$ there exist $u^* \in dom f_{\bar{x}}^*$, $v^* \in domg^*$ such that*

$$L(\bar{x}, x^*, u^*, v^*) \geq g(\bar{x}) - h(\bar{x}), \tag{23}$$

then $\bar{x}$ is a global solution of (GEP).

Proof Let $x^* \in \operatorname{dom} h^*$. By assumption, there exist $u^* \in \operatorname{dom} f_{\bar{x}}^*$, $v^* \in \operatorname{dom} g^*$ such that (23) holds. Using the same argument as in the first part of the proof of Theorem 3.2, we obtain, for all $y \in K$, that

$$\begin{aligned} g(\bar{x}) - h(\bar{x}) &\leq L(\bar{x}, x^*, u^*, v^*) = h^*(x^*) - f_{\bar{x}}^*(u^*) - g^*(v^*) - \delta_K^*(x^* - u^* - v^*) \\ &\leq h^*(x^*) - \langle x^*, y \rangle + f_{\bar{x}}(y) + g(y). \end{aligned} \tag{24}$$

Since x^* is arbitrary, we deduce from the previous inequality that for all $y \in K$,

$$g(\bar{x}) - h(\bar{x}) \leq \inf_{x^* \in \operatorname{dom} h^*} \{h^*(x^*) - \langle x^*, y \rangle\} + f_{\bar{x}}(y) + g(y), \ \forall y \in K.$$

Consequently,

$$g(\bar{x}) - h(\bar{x}) \leq f_{\bar{x}}(y) + g(y) - h(y) \ \text{ for all } y \in K,$$

i.e.,

$$g(\bar{x}) - h(\bar{x}) \le p(\bar{x}).$$

Combining this last inequality and the obvious inequality $p(\bar{x}) \le \Psi(\bar{x})$, we get $p(\bar{x}) = \Psi(\bar{x})$ and thus $\bar{x}$ is a solution of (GEP). □

The following corollary gives necessary and sufficient conditions for a point $\bar{x} \in K$ to be a solution of (GEP) in the special case when $h \equiv 0$.

Corollary 4.1 (Necessary and sufficient optimality condition for (GEP) when $h \equiv 0$) *Let $\bar{x} \in K$. For problem (GEP), assume that $h \equiv 0$ and that the closedness condition (CC) holds at $\bar{x}$. Then $\bar{x}$ is a solution of (GEP) if and only if the dual problem $(DGEP)$ has a solution, i.e., there exist $u^* \in \mathrm{dom} f_{\bar{x}}^*$, $v^* \in \mathrm{dom} g^*$ such that*

$$f_{\bar{x}}^*(u^*) + g^*(v^*) + \delta_K^*(-u^* - v^*) = -g(\bar{x}). \tag{25}$$

Proof First note that when $h \equiv 0$, any local solution of (GEP) is a global one. The necessity is a direct consequence of Theorem 4.2 when taking $h \equiv 0$ and $x^* = 0$ (since in that case $\partial h(\bar{x}) = \{0\} = \mathrm{dom} h^*$).
For the sufficiency, note that for any $y \in K$, we have

$$g(\bar{x}) = -f_{\bar{x}}^*(u^*) - g^*(v^*) - \delta_K^*(-u^* - v^*) \le f_{\bar{x}}(y) + g(y).$$

Then $\Psi(\bar{x}) \le p(\bar{x})$ and the conclusion follows. □

As it is mentioned in Remark 4.2 (see also Theorem 4.1), if (GEP) has a solution $\bar{x}$ then $v(GEP) = 0$ and if the (CC) condition holds for (GEP) at $\bar{x}$ then $v(DGEP) = 0$. The converse, however, is not true, i.e., the value of the dual problem $(DGEP)$ is zero does not imply the existence of solutions to the primal problem (GEP). This is true even for a special case of (GEP) as mentioned in [23]. In that case, however, for arbitrary small $\epsilon > 0$, problem (GEP) has ϵ-solutions as it is proven in the next theorem. But first we give a precise definition of an ϵ-solution for (GEP).

Definition 4.3 (*Definition of ϵ-solution of* (GEP)) Given $\epsilon > 0$. An element $\bar{x} \in K$ is called an *ϵ-solution* of (GEP) if

$$f(\bar{x}, y) + \Psi(y) \ge \Psi(\bar{x}) - \epsilon \quad \text{for all } y \in K.$$

The following theorem extends the results in [23] for equilibrium problems to problems of the model (GEP).

Theorem 4.5 *If the value of the dual problem $(DGEP)$ is 0, then for any $\epsilon > 0$, there is an ϵ-solution for (GEP). The converse holds provided that the closedness condition (CC) holds for all $x \in K$.*

Proof Assume that $v(DGEP) = 0$. Since $v(DGEP) \le v(P) \le 0$, we have $v(P) = 0$, which means that

$$\sup_{x \in K} [p(x) - \Psi(x)] = 0.$$

So, for any $\epsilon > 0$, there exists $\bar{x} \in K$ such that $p(\bar{x}) - \Psi(\bar{x}) \ge -\epsilon$, or, equivalently,

$$f(\bar{x}, y) + \Psi(y) \ge \Psi(\bar{x}) - \epsilon \quad \text{for all } y \in K.$$

Hence $\bar{x}$ is an ϵ-solution of (GEP).

Conversely, if for any $\epsilon > 0$, (GEP) has an ϵ-solution, then we get

$$p(\bar{x}) - \Psi(\bar{x}) \geq -\epsilon,$$

which implies $\nu(P) \geq -\epsilon$ for any $\epsilon > 0$. Hence, $\nu(P) \geq 0$. But this means that $\nu(P) = 0$. Since (CC) holds for all $x \in K$, it now follows from the previous inequality and Theorem 4.1 that $\nu(DGEP) = \nu(P) = 0$. This completes the proof. □

In the following theorem some further sufficient optimality conditions are established for problem (GEP).

Theorem 4.6 (Sufficient Optimality Conditions). *Let $\bar{x} \in K$. Consider the following statements:*

(a) *$\bar{x}$ is a solution of (GEP),*
(b) $p(x) \geq g(\bar{x}) - h(\bar{x})$ *for all $x \in K$,*
(c) $\inf\limits_{x^* \in X^*} \max\limits_{u^*, v^* \in X^*} L(x, x^*, u^*, v^*) \geq g(\bar{x}) - h(\bar{x})$ *for all $x \in K$,*
(d) *for each $x \in K$ and each $x^* \in X^*$ there exist $u^*, v^* \in X^*$ satisfying*

$$L(x, x^*, u^*, v^*) \geq g(\bar{x}) - h(\bar{x}). \tag{26}$$

The following conclusions hold:

(i) (b) *implies* (a).
(ii) *If for all $x \in K$, problem (P_x) satisfies (CC), then (b), (c), and (d) are equivalent together. Moreover, in this case each of these conditions implies (a).*

Proof (i) If (b) holds then $p(\bar{x}) \geq \Psi(\bar{x})$ and hence, $p(\bar{x}) = \Psi(\bar{x})$ which implies $\bar{x}$ is a solution of (GEP), which is (a).

(ii) Since (CC) holds at each $x \in$, the fact that (c) is equivalent to (d) follows from Proposition 3.1 (ii). On the other hand, strong duality for (P_x) (Corollary 3.5), implies that

$$p(x) = \inf_{x^* \in X^*} \left\{ \max_{u^*, v^* \in X^*} L(x, x^*, u^*, v^*) \right\}.$$

This shows that (c) is equivalent to (b) and that (ii) holds. □

Another dual scheme has been recently introduced in [18] for problem (GEP) when $h \equiv 0$. It is easy to adapt it to the general case where h is not necessarily the zero function. More precisely, this dual scheme becomes:

$$(DLGEP) \qquad \inf_{x^* \in X^*} \sup_{u^*, v^* \in X^*} \tilde{L}(x^*, u^*, v^*),$$

where $\tilde{L}(x^*, u^*, v^*) = \inf_{x \in K} L(x, x^*, u^*, v^*)$.

The value of the dual problem $(DLGEP)$ is denoted $\nu(DLGEP)$. When $h \equiv 0$ we find again the dual introduced in [18]. The corresponding weak duality property is given in the next theorem.

Theorem 4.7 (Weak duality) *Let $x \in K$. Then for any $y \in K$, it holds*

$$\nu(DLGEP) = \inf_{x^* \in X^*} \sup_{u^*, v^* \in X^*} \tilde{L}(x^*, u^*, v^*) \leq f_x(y) + g(y) - h(y), \tag{27}$$

$$\nu(DLGEP) = \inf_{x^* \in X^*} \sup_{u^*, v^* \in X^*} \tilde{L}(x^*, u^*, v^*) \leq p(x) \leq g(x) - h(x). \tag{28}$$

In particular, one has

$$\nu(DLGEP) = \inf_{x^* \in X^*} \sup_{u^*, v^* \in X^*} \tilde{L}(x^*, u^*, v^*) \leq \inf_{x \in K} p(x). \tag{29}$$

Proof Let $x^*, u^*, v^* \in X^*$ and let $x \in K$. By definition of conjugate functions, it is clear that for any $y \in K$,

$$\begin{aligned} L(x, x^*, u^*, v^*) &\leq h^*(x^*) + \langle -u^*, y\rangle + f_x(y) + \langle -v^*, y\rangle + g(y) + \langle -x^* + u^* + v^*, y\rangle \\ &\quad + \delta_K(y) \\ &\leq h^*(x^*) - \langle x^*, y\rangle + f_x(y) + g(y). \end{aligned} \tag{30}$$

This yields for any $y \in K$,

$$\tilde{L}(x^*, u^*, v^*) \leq h^*(x^*) - \langle x^*, y\rangle + \inf_{x \in X}[f_x(y)] + g(y) \leq h^*(x^*) - \langle x^*, y\rangle + f_x(y) + g(y),$$

which implies for any $y \in K$ (note that $h^{**}(y) = h(y)$),

$$\begin{aligned} \inf_{x^* \in X^*} \sup_{u^*, v^* \in X^*} \tilde{L}(x^*, u^*, v^*) &\leq \inf_{x^* \in X^*} [h^*(x^*) - \langle x^*, y\rangle] + f_x(y) + g(y) \\ &= -h^{**}(y) + f_x(y) + g(y) \\ &= f_x(y) + g(y) - h(y). \end{aligned}$$

The first assertion is proven. The second assertion follows from the arbitrariness of $y \in K$. The weak duality (29) is immediate from (28). The proof is complete. □

Theorem 4.8 (Strong duality). *Assume that* $\alpha := \inf_{x \in K} p(x) \in \mathbb{R}$. *Assume further that for each* $x \in K$, *problem* (P_x) *satisfies* (CC). *Then it holds*

(i) $\nu(DLGEP) = \inf_{x \in K} p(x)$,
(ii) $\nu(DLGEP) = \inf_{x \in X} \inf_{x^* \in X^*} \sup_{u^*, v^* \in X^*} L(x, x^*, u^*, v^*)$.

Proof (i). By assumption, for each $x \in K$, $p(x) \geq \alpha$ which means that for each $x \in K$,

$$y \in K \implies f_x(y) + g(y) - h(y) \geq \alpha.$$

It follows from Farkas lemma (Corollary 3.4) that for any $x^* \in X^*$, there exist $u^*, v^* \in X^*$ such that

$$L(x, x^*, u^*, v^*) = h^*(x^*) - f_x^*(u^*) - g^*(v^*) - \delta_K^*(x^* - u^* - v^*) \geq \alpha.$$

The last inequality yields

$$\inf_{x^* \in X^*} \sup_{u^*, v^* \in X^*} \tilde{L}(x^*, u^*, v^*) \geq \alpha = \inf_{x \in K} p(x).$$

So (i) follows from this and the weak duality property (Theorem 4.7). Finally part (ii) is obtained from (i) and Corollary 3.5 (ii). □

Corollary 4.2 *Under the assumptions of Theorem 4.8, if* $\Psi \equiv 0$, *then*

$$\nu(DLGEP) \leq \nu(DGEP) \leq \nu(P).$$

Theorem 4.9 *Let* $\bar{x} \in K$. *If for all* $x \in K$, *problem* (P_x) *satisfies* (CC) *and* $\nu(DLGEP) = g(\bar{x}) - h(\bar{x})$, *then* $\bar{x}$ *is a solution of* (GEP).

Proof By the strong duality property (Theorem 4.8), we have

$$v(DLGEP) = \inf_{x \in K} p(x) = g(\bar{x}) - h(\bar{x}).$$

Then the conclusion follows from Theorem 4.6 (i). □

5 Gap functions for (GEP)

Gap functions are useful for solving variational inequalities and equilibrium problems (see, for example, the references [1, 2, 16, 24]). In this section we study gap functions associated with (GEP). They are based on the duality results developed in the previous sections. Furthermore, when (GEP) represents a variational inequality problem or an equilibrium problem, these gap functions coincide with the gap functions given in [1, 2]. Let us first recall the definition of a gap function for (GEP).

Definition 5.1 (*Gap function*) A function $q : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a *gap function* for (GEP) if the following conditions hold:

(i) $q(x) \geq 0$ for all $x \in K$,
(ii) $\bar{x} \in K$ is a solution of (GEP) if and only if $q(\bar{x}) = 0$.

Let

$$\bar{\gamma}(x) := -\inf_{y \in K} [f(x, y) + \Psi(y) - \Psi(x)]. \tag{31}$$

By Lemma 3.1, x^* is a solution of (GEP) if and only if $\bar{\gamma}(x^*) = 0$. Since $\bar{\gamma}(x) \geq 0$ for all $x \in K$, it is clear that $\bar{\gamma}$ is a gap function for (GEP).

When (GEP) represents the standard variational inequality problem: Find $x \in K$ such that

$$\langle F(x), y - x \rangle \geq 0 \text{ for all } y \in K,$$

the gap function defined by (31) reduces to

$$\bar{\gamma}(x) = \sup_{y \in K} \langle F(x), x - y \rangle$$

which is the gap function introduced for this problem by Auslender [4]. Here F is an operator from X to its dual X^*.

Thanks to the duality theory developed previously we can introduce another class of gap functions for (GEP). More precisely, for all $x \in K$, we define

$$\gamma(x) := -\left[\inf_{x^* \in X^*} \sup_{u^*, v^* \in X^*} L(x, x^*, u^*, v^*) \right] + g(x) - h(x). \tag{32}$$

This function γ is called "the dual gap function" associated with (GEP). This name is justified in the proof of the next theorem.

Theorem 5.1 (Gap function for (GEP)). *Assume that problem (P_x) satisfies the condition (CC) for each $x \in K$. Then γ is a gap function for (GEP).*

Proof It follows from the weak duality for (GEP) (Theorem 4.1) that $\gamma(x) \geq 0$ for all $x \in K$. On the other hand, by Lemma 3.1, $\bar{x}$ solves (GEP) if and only if $p(\bar{x}) = \Psi(\bar{x})$. Since the condition (CC) holds at $\bar{x}$, the strong duality property for $(P_{\bar{x}})$ gives

$$p(\bar{x}) = \inf_{x^* \in X^*} \sup_{u^*, v^* \in X^*} L(\bar{x}, x^*, u^*, v^*).$$

So, $\bar{x}$ solves (GEP) if and only if $\gamma(\bar{x}) = \Psi(\bar{x}) - p(\bar{x}) = 0$. □

The gap function γ defined in (32) collapses to the ones constructed in [1] and [2] for the variational inequality problem and for the equilibrium problem, respectively.

6 Convex and DC optimization problems

In this section we apply the duality and optimality conditions for (GEP) to convex and DC programs. We find again several duality results for these classes of problems established in [13, 14]. Better still, we get some new optimality conditions for these problems.

6.1 Convex optimization problems

First let us consider the following convex optimization problem with cone-convex constraints

$$(PC) \begin{cases} \min \ \Psi(y) \\ \text{s.t.} \quad y \in C, \ k(y) \in -S. \end{cases}$$

Here, X and Z are locally convex topological spaces, C is a closed convex subset of X, S is a closed convex cone of Z, $\Psi : X \to \mathbb{R} \cup \{+\infty\}$ is a proper, l.s.c., and convex function while $k : X \to Z$ is a S-convex and continuous mapping. For problem (PC), we assume that its feasible set $K := \{y \in C \mid k(y) \in -S\}$ is non-empty. Let

$$S^+ := \{z^* \in Z^* \mid \langle z^*, s\rangle \geq 0 \quad \text{for all } s \in S\},$$

be the positive dual cone of S.

Note that problem (PC) can be rewritten as follows:

$$(GEP1) \begin{cases} \text{Find } x \in K \text{ such that} \\ \Psi(y) \geq \Psi(x) \ \text{ for all } y \in K. \end{cases}$$

This is a special case of (GEP) with $f \equiv 0$, $g(y) = \Psi(y)$ for all $y \in X$, and $h \equiv 0$.

Observe that when $h \equiv 0$, we have $h^*(0) = 0, h^*(x^*) = +\infty$ if $x^* \neq 0$, and thus epi $h^* = \{0\} \times [0, +\infty)$. The same holds for $f_x \equiv 0$, i.e., epi $f_x^* = \{0\} \times [0, +\infty)$. Therefore, for $(GEP1)$, the function L does not depend on the variables x, x^*, and v^* (in fact, $x^* = v^* = 0$) and collapses to

$$L(u^*) = -\Psi^*(-u^*) - \delta_K^*(u^*).$$

In that case the two dual problems $(DGEP)$ and $(DLGEP)$ are equivalent and can be written under the form

$$(D1) \qquad \sup_{u^* \in X^*} L(u^*).$$

Furthermore the two duality theories developed in Sect. 4 can be applied to problem $(D1)$.

On the other side note that for each $\lambda \in S^+$, the function λk defined by $\lambda k(x) := \langle \lambda, k(x)\rangle$ for all $x \in X$, is continuous. By Lemma 2 in [21] (see also [20]),

$$\text{epi } \delta_K^* = \text{cl}\left\{\bigcup_{\lambda \in S^+} \text{epi } (\lambda k)^* + \text{epi } \delta_C^*\right\}. \tag{33}$$

As a consequence of the strong duality property, (Theorem 4.8), we obtain the following Fenchel and Fenchel-Lagrange duality results for problem (PC). These results were recently established in [8, 13, 14].

Theorem 6.1 (Fenchel Duality and Fenchel-Lagrange Duality). *Suppose that* $v(PC) \in \mathbb{R}$.

(i) *If epi* $\Psi^* +$ *epi* δ_K^* *is weak***-closed, then Fenchel duality holds, i.e.,*

$$v(PC) = \sup_{u^* \in X^*} [-\Psi^*(-u^*) - \delta_K^*(u^*)]. \tag{34}$$

(ii) *If epi* $\Psi^* + \bigcup_{\lambda \in S^+}$ *epi* $(\lambda k)^* +$ *epi* δ_C^* *is weak***-closed, then the Fenchel-Lagrange duality holds, i.e.,*

$$v(PC) = \max_{\lambda \in S^+;\ u^*, v^* \in X^*} [-\Psi^*(-u^*) - (\lambda k)^*(v^*) - \delta_C^*(u^* - v^*)], \tag{35}$$

(*the "sup" in the right hand-side of* (35) *is attained at some* $\lambda \in S^+$*, and* $u^*, v^* \in X^*$).

Proof (i) Note that the dual problem $(D1)$ can be written in the form

$$(D1a) \qquad \sup_{u^* \in X^*} [-\Psi^*(-u^*) - \delta_K^*(u^*)].$$

Then the Fenchel duality (i) follows from Theorem 4.8.

(ii) The conclusion (ii) can be proved using an argument similar to the one in the proof of Theorem 3.2 (ii), where the set epi $\Psi^* + \bigcup_{\lambda \in S^+}$ epi $(\lambda k)^* +$ epi δ_C^* plays the role of epi $F^* +$ epi $G^* +$ epi δ_K^* and $H \equiv 0$. Also, by the proof of Theorem 3.2 (ii), the "sup" in the right hand-side of (35) is attained at some $\lambda \in S^+$, and $u^*, v^* \in X^*$). □

It is worth mentioning that the Fenchel duality result given in Theorem 6.1 (i) is related to the Fenchel dual of the problem (PC) when written under the form $\inf_{x \in X}[\Psi(x) + \delta_K(x)]$. Similar Fenchel duality results related to the more general convex problem $\inf_{x \in X}[p(x) + q(x)]$ were given in [3] under another regularity condition which is strictly stronger than the one in Theorem 6.1, and was given in [8] under the same type of qualification condition. On the other hand, the Fenchel-Lagrange duality result given in Theorem 6.1 (ii) was proved in [13, 14] under the same constraint qualification and in [7] under another constraint qualification.

We now show that the strong Lagrange duality for (PC) can be derived from Theorem 4.8. This result was established in several papers such as [11, 14, 20].

Theorem 6.2 (Lagrange Duality) [11, 14, 20]. *Suppose that*

$$\textit{epi } \Psi^* + \bigcup_{\lambda \in S^+} \textit{epi } (\lambda k)^* + \textit{epi } \delta_C^*$$

is weak**-closed. Then strong Lagrange duality holds for* (PC)*, i.e.,*

$$\inf_{y \in C, k(y) \in -S} \Psi(y) = \sup_{\lambda \in S^+} \inf_{y \in C} [\Psi(y) + (\lambda k)(y)]. \tag{36}$$

Proof Using a same argument as in the proof of Theorem 6.1 (ii), we obtain

$$v(D) = \sup_{\lambda \in S^+; u^*, v^* \in X^*} [-\Psi^*(-u^*) - (\lambda k)^*(v^*) - \delta_C^*(u^* - v^*)].$$

Moreover, it is also easy to see that

$$-\Psi^*(-u^*) - (\lambda k)^*(v^*) - \delta_C^*(u^* - v^*)] \leq \Psi(y) + (\lambda k)(y), \ \forall y \in C,$$

which implies

$$-\Psi^*(-u^*) - (\lambda k)^*(v^*) - \delta_C^*(u^* - v^*)] \leq \inf_{y \in C} [\Psi(y) + (\lambda k)(y)] \leq \nu(PC)$$

(the last inequality holds by weak duality). The equality (36) readily follows from Theorem 4.8. □

We now turn to the application of optimality conditions for (GEP) (see Theorem 4.6) to (PC). For the sake of simplicity, we consider the case where $C = X$ and as usual, let $K = k^{-1}(-S)$.

Theorem 6.3 *For the problem (PC), assume that $C = X$ and that $\bar{x} \in K$. Assume further that the set epi $\Psi^* + \bigcup_{\lambda \in S^+}$ epi $(\lambda k)^*$ is weak*-closed. The following statements are equivalent:*

(i) *$\bar{x}$ is a solution of (PC),*
(ii) *There exist $\lambda \in S^+$, $u^* \in X^*$ such that*

$$-\Psi^*(-u^*) - (\lambda k)^*(u^*) \geq \Psi(\bar{x}),$$

(iii) *There exists $\lambda \in S^+$ such that*

$$\Psi(y) + (\lambda k)(y) \geq \Psi(\bar{x}) \quad \textit{for all } y \in X.$$

Proof We first give some observations gathered from Theorems 6.1 and 6.2.
Under the given assumptions and $\nu(PC) < +\infty$, by Theorem 6.1 we get
(α) there exist $\lambda \in S^+$ and $u^* \in X^*$ such that

$$\nu(PC) = -\Psi^*(-u^*) - (\lambda k)^*(u^*).$$

On the other hand, by Theorem 6.2, we get
(β) there exists $\lambda \in S^+$ such that

$$\nu(PC) = \inf_{y \in C} [\Psi(y) + (\lambda k)(y)].$$

Hence (α) and (β) are equivalent.
From the previous observations, (ii) and (iii) are equivalent and each of them implies (i) by Theorem 4.6. The fact that (i) implies (ii) follows from Theorem 4.3 (see also its proof). The proof is complete. □

It is worth mentioning that the optimality condition given in (ii) is new while the one in (iii) may be obtained by using the new Farkas-type result established recently in [14].

6.2 DC optimization problems

In this subsection we consider DC optimization problems with convex constraints, i.e., problems of the form (PC) with $\Psi(x) = g(x) - h(x)$ being a DC function. More precisely, we consider the problem

$$(PDC)\begin{cases} \min\ \Psi(y) := g(y) - h(y) \\ \text{s.t.}\quad y \in C,\ k(y) \in -S, \end{cases}$$

where X, C, S, and Z are as before and $g, h : X \longrightarrow \mathbb{R}\cup\{+\infty\}$ are proper, l.s.c., and convex functions. Let K be the set defined as before, i.e., $K := \{y \in C \mid k(y) \in -S\}$.

It is obvious that (PDC) can be rewritten under the form of (GEP), say $(GEP2)$, as for problem (PC) with $f \equiv 0$. In that case, the function L is independent from the variables x and v^*.

$$L(x^*, u^*) = h^*(x^*) - g^*(u^*) - \delta_K^*(x^* - u^*).$$

Consequently the two dual problems $(DGEP)$ and $(DLGEP)$ are equivalent and can be written under the form

$$(D2) \qquad \inf_{x^*\in X^*} \sup_{u^*\in X^*} L(x^*, u^*).$$

Furthermore the two duality theories developed in Section 4 can be applied to problem $(D2)$. Problem $(D2)$ is the Toland-Fenchel-Lagrange dual problem introduced in [14, 22]. The strong duality for (GEP) (Theorem 4.8) entails the following strong Toland-Fenchel-Lagrange duality result for (PDC) which is the one established recently in [13, 14]. The two following theorems give a duality result and an optimality condition for (PDC). The first one was introduced in [7, 14] while the second one is a new optimality condition for (PDC). These theorems can be proved by the same arguments as those of Theorems 6.1 and 6.3 and will be omitted.

Theorem 6.4 [13, 14]. *If epi Ψ^* + epi δ_K^* is weak*-closed, then*

$$v(PDC) = \inf_{x^*\in X^*} \sup_{u^*\in X^*} [h^*(x^*) - g^*(u^*) - \delta_K^*(x^* - u^*)].$$

Proposition 6.1 (Optimality Condition for (PDC)). *Assume that $C = X$, g is continuous at some point in $K = k^{-1}(-S)$, and that the set $\bigcup_{\lambda\in S^+}$ epi $(\lambda k)^*$ is weak*-closed. The following statements are equivalent:*

(i) *$\bar{x} \in K$ is a solution of (PDC),*
(ii) *For each $x^* \in X^*$, there exist $\lambda \in S^+$, $u^* \in X^*$ such that*

$$h^*(x^*) - g^*(u^*) - (\lambda k)^*(x^* - u^*) \ge g(\bar{x}) - h(\bar{x}).$$

7 Equilibrium problems

Consider the general equilibrium problem

$$(EP)\ \begin{cases} \text{Find } x \in K \text{ such that} \\ f(x, y) + g(y) \ge g(x) \text{ for all } y \in K, \end{cases}$$

where $f : X \times X \longrightarrow \mathbb{R}\cup\{+\infty\}$ is such that, for each $x \in K$, $f_x(.)$ is a proper, l.s.c., and convex function, $g : X \to \mathbb{R}\cup\{+\infty\}$ is a proper, l.s.c., and convex function and K is a closed convex subset of X. This problem is a particular case of problem (GEP) when $\Psi \equiv g$ and $h \equiv 0$. In that case, for each $x \in K$, the problem (P_x) becomes $\inf_{y\in K}(f_x(y)+g(y))$, and the condition (CC) for (P_x) can be expressed as: epi f_x^*+epi g^*+epi δ_K^* is weak*-closed. Furthermore, since epi $h^* = \{0\}\times[0,+\infty)$, the function L only depends on $x \in X$, $u^*, v^* \in X^*$ and can be written as

$$L(x,u^*,v^*) = -f_x^*(u^*) - g^*(v^*) - \delta_K^*(-u^*-v^*). \tag{37}$$

The function L depending on x, the two dual problems $(DGEP)$ and $(DLGEP)$ are different.

In this section, we consider two cases. The first one corresponds to $K = X$ and has been studied from the duality point of view particularly in [18]. In the second one, the function $g \equiv 0$ and K is some closed convex subset of X. This is the equilibrium problem introduced by Blum and Oettli in [6] and for which a theory of duality has been developed in [23].

Let us start with the first case where $K = X$. Then, the support function $\delta_K^*(-u^*-v^*) = 0$ if $v^* = -u^*$ and $+\infty$ otherwise. Consequently, the function L only depends on $x \in X$ and $u^* \in X^*$ and has the form

$$L(x,u^*) = -f_x^*(u^*) - g^*(-u^*). \tag{38}$$

With this function L, the general dual problem $(DGEP)$ becomes

$$(D3) \qquad \max_{x\in X}\ \sup_{u^*\in X^*}\ \left[L(x,u^*) - g(x)\right].$$

By the weak duality theorem (Theorem 4.1), we have that $v(D3) \le \sup_{x\in X}\ [p(x) - g(x)] \le 0$. Furthermore, using (37) and (38), it follows from Corollary 4.1 that $\bar{x} \in X$ is a solution of problem (EP) if and only if there exists $u^* \in X^*$, satisfying $L(x,u^*) \ge g(\bar{x})$, provided that the closedness condition (CC) holds at $\bar{x}$. So we find again the optimality conditions obtained in [18] under a similar qualification of constraints. If we consider the general dual problem $(DLGEP)$, we obtain the following dual problem

$$(D4) \qquad \sup_{u^*\in X^*}\ \inf_{x\in X}\ L(x,u^*).$$

By the weak duality theorem (Theorem 4.7), we have that $v(D4) \le \inf_{x\in X}\ p(x)$. Since $p(x) \le g(x)$ for all $x \in X$, we obtain that $v(D4) \le \inf_{x\in X} g(x)$. So we find again the weak duality theorem obtained in [18] under a similar qualification of constraints. However, in addition to this result, we can derive from Theorem 4.8, the following strong duality property.

Corollary 7.1 *The optimal value of problem* $(D4)$ *is such that* $v(D4) = \inf_{x\in X}\ p(x)$ *provided that* $\inf_{x\in X}\ p(x)$ *is finite and that, for every* $x \in X$*, the subset epi* f_x^**+epi* g^* *is weak*-closed in* X^*.

Finally let us mention that when $K = X$, the gap function (32) becomes:

$$\gamma_1(x) := \inf_{u^*\in X^*}[f_x^*(u^*) + g^*(-u^*)] + g(x). \tag{39}$$

Remark 7.1 In [18], the function $f(x,y)$ is decomposed into the sum of two functions $F(x,y) + \varphi(x,y)$ to recover more easily the various equilibrium problems considered in the literature. In [18], this decomposition is not used to develop the duality theory. However the authors of [18] work with the function $f(x,\cdot) = F(x,\cdot) + \varphi(x,\cdot)$ that is assumed to be proper, l.s.c., and convex and so, our method is applicable in this case.

We now examine the important particular case of variational inequality problems in the case where $K = X$. More precisely, we consider the variational inequality problem of the form

$$(VI) \quad \begin{cases} \text{Find } x \in X \text{ such that} \\ \langle A(x), y - x\rangle + g(y) \geq g(x) \quad \text{for all } y \in X, \end{cases}$$

where A is an operator from X to its dual X^* and $g : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a proper, l.s.c., and convex function. This model was first studied in [25], and then in [5, 18]. In this section we show that our dual problem defined in Sect. 4 reduces to the dual problem and the results established by Mosco in [25].

Let $f(x, y) = \langle A(x), y-x\rangle$. Then (VI) is a particular case of (GEP) where $h \equiv 0$, $K = X$, and $\Psi = g$. To show that our dual problem for (VI), considered as a representation of (GEP), is equivalent to the dual introduced by Mosco in [25], we proceed as follows: First we observe that for each $x \in X$,

$$f_x^*(u^*) = \begin{cases} \langle A(x), x\rangle & \text{if } u^* = A(x), \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover, $\mathrm{dom} h^* = \{0\}$ with $h^*(0) = 0$. The same holds for δ_K, i.e., $\mathrm{dom}\delta_K^* = \{0\}$ and $\delta_K^*(0) = 0$. So, the function L (defined by (16)) in this case only depends on x and $u^* \in \mathrm{dom} f_x^*$,

$$L(x, u^*) = -f_x^*(u^*) - g^*(-u^*) \tag{40}$$

$$= \begin{cases} -\langle A(x), x\rangle - g^*(-A(x)), & \text{if } u^* = A(x), -u^* \in \mathrm{dom} g^*, \\ -\infty & \text{otherwise .} \end{cases} \tag{41}$$

The dual problem for (VI), called (DVI), is

$$(DVI) \qquad \max_{x\in X} \max_{u^*\in X^*, u^*=A(x)} \left[-\langle u^*, x\rangle - g^*(-u^*) - g(x)\right].$$

Now let $x \in X$. Since f_x is continuous on X, it follows from (2) that epi f_x^* + epi g^* is weak*-closed. So condition (CC) is satisfied for every $x \in X$ and, by the strong duality theorem (Theorem 4.1), the optimal value of problem (DVI) is equal to zero. Consequently, finding a solution to problem (DVI) consists in finding $\bar{x} \in X$ and $u^* \in X^*$ such that $u^* = A(\bar{x})$ and

$$-\langle u^*, \bar{x}\rangle - g^*(-u^*) - g(\bar{x}) = 0.$$

In other terms, problem (DVI) is equivalent to the following one: Find $\bar{x} \in X$ and $u^* \in X^*$ such that $u^* = A(\bar{x})$ and $\bar{x} \in \partial g^*(-u^*)$, i.e., such that

$$\begin{cases} -\langle \bar{x},\ v^* + u^*\rangle + g^*(v^*) \geq g^*(-u^*) \quad \text{for all } v^* \in X^*, \\ u^* = A(\bar{x}). \end{cases}$$

So we find again the dual proposed by Bigi, Castellani and Kassay (see (10) in [5]). In particular, when A is an injective mapping, $\bar{x} = A^{-1}(u^*)$ is the unique solution associated with u^* and problem (DVI) collapses into the variational inequality:

$$\text{Find } u^* \in X^* \text{ such that } \langle -A^{-1}(u^*),\ v^* + u^*\rangle + g^*(v^*) \geq g^*(-u^*) \quad \text{for all } v^* \in X^*$$

introduced by Mosco in [25] as the dual of (VI). Furthermore, the duality theorem obtained by Mosco in [25] is a direct consequence of Corollary 4.1.

Finally it is easy to see that, for problem (VIP), the gap function (39) becomes:

$$\gamma_1(x) = \inf_{u^* \in X^*} \langle A(x),\ x\rangle + g^*(-A(x)) + g(x).$$

This gap function was introduced in [1] for solving problem (VI).

Now we consider the second case: $g \equiv 0$ and K is some closed convex subset of X. In that case, problem (P_x) becomes $\inf_{y\in K} f_x(y)$, and since epi $g^* = \{0\} \times [0, +\infty)$, the condition (CC) for (P_x) becomes

$$\text{epi } f_x^* + \text{epi } \delta_K^* \quad \text{is} \quad \text{weak}^*\text{-closed in } X^*.$$

Several sufficient conditions to obtain condition (CC) were given in [8, 14]. In particular, (P_x) satisfies (CC) if either f_x is continuous at some point in K or int $K \neq \emptyset$ and $\text{dom} f_x \cap \text{int} K \neq \emptyset$ (see [23]).

Since $g \equiv 0$, the conjugate $g^*(v^*) = 0$ if $v^* = 0$ and $+\infty$, otherwise. Hence, the function L only depends on $x \in K$ and $u^* \in X$ and can be written as

$$L(x, u^*) = -f_x^*(u^*) - \delta_K^*(-u^*). \tag{42}$$

Adopting the notation used in [23], we set $i_K(u^*) := \inf_{y\in K}\langle u^*, y\rangle$. Then, $\delta_K^*(-u^*) = -i_K(u^*)$ and

$$\begin{aligned}\sup_{x\in K}\ \sup_{u^*, v^* \in X^*} L(x, u^*) &= \sup_{x\in K}\ \sup_{u^*\in X^*} [-f_x^*(u^*) - \delta_K^*(-u^*)]\\ &= \sup_{u^*\in X^*} [i_K(u^*) - \inf_{x\in K} f_x^*(u^*)].\end{aligned}$$

The dual problem of (EP) is then

$$(D5) \qquad\qquad \sup_{u^*\in X^*} [i_K(u^*) - \inf_{x\in K} f_x^*(u^*)].$$

So, in this particular case, our dual problem $(DGEP)$ coincides with the dual problem $(D5)$ introduced by Martinez-Legaz and al. in [23]. The weak and strong duality properties given in Theorem 4.1, become: $v(D5) \le \sup_{x\in K} p(x)$ and $v(D5) = \sup_{x\in K} p(x)$ when (P_x) satisfies (CC) for each $x \in K$. Furthermore, since $p(x) \le 0$, we also obtain that $v(D5) \le 0$. For this problem, and with L defined by (42), the optimality conditions given in Corollary 4.1, become: if condition (CC) is satisfied at $\bar{x}$, then $\bar{x} \in X$ is a solution of problem (EP) if and only if there exists $u^* \in X^*$, satisfying $L(\bar{x}, u^*) \ge 0$, i.e., $i_K(u^*) - f_{\bar{x}}^*(u^*) \ge 0$. This condition allows us to derive the following proposition closely related to Theorem 3.1 in [23].

Proposition 7.1 *Let $\bar{x} \in K$. Assume that epi $f_{\bar{x}}^*$+epi δ_K^* is weak*-closed. Then the following statements are equivalent:*

(i) *$\bar{x} \in K$ is a solution of (EP),*
(ii) *There exists $u^* \in X^*$ such that $f_{\bar{x}}^*(u^*) = i_K(u^*)$.*
Furthermore, if either (i) or (ii) holds, then $i_K(u^*) = \inf_{x\in K} f_x^*(u^*)$.

Proof By Corollary 4.1, $\bar{x} \in K$ is a solution of (EP) if and only if there exists $u^* \in X^*$ such that $L(\bar{x}, u^*) \ge 0$, or, equivalently,

$$f_{\bar{x}}^*(u^*) \le i_K(u^*).$$

On the other hand, it follows from the definition of the conjugate function that, for any $x, y \in K$, one has $\langle u^*, y\rangle - f_x^*(u^*) \leq f(x, y)$. So, taking successively the infimum on y and the supremum on x , we obtain

$$i_K(u^*) - f_x^*(u^*) \leq \inf_{y \in K} f(x, y) \leq 0 \quad \text{for all } x \in K,$$

and

$$i_K(u^*) - \inf_{x \in K} f_x^*(u^*) \leq 0.$$

Consequently, $i_K(u^*) = f_{\bar{x}}^*(u^*) = \inf_{x \in K} \; f_x^*(u^*)$. The proof is complete. □

Remark 7.2 It is worth mentioning that this result was established in [23] under the constraint qualification:

"for every $x \in K$, there exists $y_x \in K$ such that $f(x, y_x) < +\infty$, and either $y_x \in intK$ or f_x is continuous at y_x."

It is easy to see that this condition implies that $0 \in \operatorname{int}(\operatorname{dom} f_x - K)$ which, in turn, entails that epi $f_{\bar{x}}^*$ + epi δ_K^* is weak*-closed (see [8, 20]). This means that Proposition 7.1 improves the result in [23].

To end this section we present a gap function for (EP). The next result is a consequence of Theorem 5.1. It covers the corresponding result established in [2] for the special case where $g \equiv 0$.

Proposition 7.2 *Suppose that for any $x \in K$, the set epi f_x^* + epi δ_K^* is weak*-closed. Then the function γ_2 defined by*

$$\gamma_2(x) := \inf_{u^* \in X^*} [f_x^*(u^*) + \delta_K^*(-u^*)] \tag{43}$$

is a gap function for (EP).

Proof Note that (EP) is a special case of (GEP) with $g = h \equiv 0$. The function L only depends on x, u^*, and can be expressed as

$$L(x, u^*) = -f_x^*(u^*) - \delta_K^*(-u^*).$$

In this case, the gap function γ defined in (32) collapses to γ_2 defined by (43). The conclusion follows from Theorem 5.1. □

Acknowledgement The first author wishes to thank the University of Namur for financial support in the form of a FUNDP Research Scholarship

References

1. Antangerel, L., Bot, R.I., Wanka, G.: On the construction of gap functions for variational inequalities via conjugate duality. Asia-Pac. J. Oper. Res. **24**, 353–371 (2007)
2. Antangerel, L., Bot, R.I., Wanka, G.: On gap functions for equilibrium problems via Fenchel duality. Pac. J. Optim. **2**, 667–678 (2006)
3. Attouch, H., Brezis, H.: Duality for the sum of convex functions in general Banach spaces. In: Barroso, J.A. (ed.) Aspects of Mathematics and its Application, pp. 125–133. Elsevier, Amsterdam, The Netherlands (1986)
4. Auslender, A.: Optimisation. Méthodes Numériques. Masson, Paris (1976)
5. Bigi, G., Castellani, M., Kassay, G.: A dual view of equilibrium problems. J. Math. Anal. Appl. **342**, 17–26 (2008)

6. Blum, E., Oettli, W.: From optimization and variational inequality to equilibrium problems. Math. Stud. **63**, 127–149 (1994)
7. Bot, R.I., Wanka, G.: A weaker regularity condition for subdifferential calculus and Fenchel duality in infinite dimensional spaces. Nonlinear Anal, to appear.
8. Burachik, R.S., Jeyakumar, V.: A new geometric condition for Fenchel's duality in infinite dimensional spaces. Math. Program. **104(B)**, 229–233 (2005)
9. Burachik, R.S., Jeyakumar, V.: A dual condition for the convex subdifferential sum formula with applications. J. Convex Anal. **12**, 279–290 (2005)
10. Dinh, N., Goberna, M.A., López, M.A., Son, T.Q.: New Farkas-type results with applications to convex infinite programming. ESAIM: Control Optim. Cal. Var. **13**, 580–597 (2007)
11. Dinh, N., Jeyakumar, V., Lee, G.M.: Sequential Lagrangian conditions for convex programs with applications to semi-definite programming. J. Optim. Theory Appl. **125**, 85–112 (2005)
12. Dinh, N., Mordukhovich, B.S., Nghia, T.T.A.: Subdifferentials of value functions and optimality conditions for some classes of DC and bilevel infinite and semi-infinite programs. Research Report # 4, Department of Mathematics, Wayne State University, Detroit, Michigan (2008) (to appear in Math. Program.)
13. Dinh, N., Nghia, T.T.A., Vallet, G.: A closedness condition and its applications to DC programs with convex constraints. Optimization, 1-20, iFirst (2008) doi:10.1080/02331930801951348 First Published on: 31 March 2008 http://www.informaworld.com/smpp/title~content=g770174694~db=all?stem=3#messages
14. Dinh, N., Vallet, G., Nghia, T.T.A.: Farkas-type results and duality for DC programs with convex constraints. J. Convex Anal. **15**, 235–262 (2008)
15. Fang, D.H., Li, C., Ng, K.F.: Constraint qualifications for extended Farkas' lemmas and Lagrangian dualities in convex infinite programming (Submitted).
16. Fukushima, M.: A class of gap functions for quasi-variational inequality problems (Preprint)
17. Hiriart-Urruty, J.B.: From convex optimization to non-convex optimization necessary and sufficient conditions for global optimality. In: Gilbert, R.P., Panagiotopoulos, P.D., Pardalos, P.M. (eds.) From Convexity to Non-convexity, pp. 219–239. Kluwer, London (2001)
18. Jacinto, F.M.O., Scheimberg, S.: Duality for generalized equilibrium problems. Optimization **57**, 795–805 (2008)
19. Jeyakumar, V.: Asymptotic dual conditions characterizing optimality for convex programs. J. Optim. Theory Appl. **93**, 153–165 (1997)
20. Jeyakumar, V., Dinh, N., Lee, G.M.: A new closed cone constraint qualification for convex optimization. Applied Mathematics research report AMR04/8, UNSW, Sydney, Australia (2004).
21. Jeyakumar, V., Wu, Z.Y., Lee, G.M., Dinh, N.: Liberating the subgradient optimality conditions from constraint qualifications. J. Glob. Optim. **34**, 127–137 (2006)
22. Laghdir, M.: Optimality conditions and Toland's duality for a non-convex minimization problem. Matematicki Vesnik **55**, 21–30 (2003)
23. Martinez-Legaz, J.E., Sosa, W.: Duality for equilibrium problems. J. Glob. Optim. **25**, 311–319 (2006)
24. Mastroeni, G.: Gap functions for equilibrium. J. Glob. Optim. **27**, 411–426 (2003)
25. Mosco, U.: Dual variational inequalities. J. Math. Anal. Appl. **40**, 202–206 (1972)
26. Muu, L.D., Nguyen, V.H., Quy, N.V.: Nash-Cournot oligopolistic market equilibrium models with concave cost functions. J. Glob. Optim. **41**, 351–364 (2008)
27. Toland, J.F.: Duality in non-convex optimization. J. Math. Anal. Appl. **66**, 399–415 (1978)
28. Zalinescu, C.: Convex Analysis in General Vector Spaces. World Scientific, Singapore (2002)